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IDENTITIES AND RELATIONS RELATED TO COMBINATORIAL NUMBERS AND POLYNOMIALS

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Abstract. This paper presents some new families of special numbers and polynomials including the Euler numbers and polynomials, the Stirling numbers of the second kind, the central factorial numbers and the array polynomials. We give some properties of these numbers and polynomials with their generating functions. Finally, by using these generating functions with their functional equations, we derive some identities and relations realeted to these special numbers and polynomials.

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1. Introduction

In order to give our paper new results including identities and relations which are associated with combinatorial numbers, the .array polynomials, the first and second kind Euler numbers and polynomials and the central factorial numbers of the first kind, we need the following definitions, generating functions an relations.

The first kind Apostol-Euler polynomials of order-higher are defined by means of the following generating function:

$$F_{P1}(t,x;k,\lambda)=\left(\frac{2}{\lambda e^t+1}\right)^k e^{tx}=\sum_{n=0}^{\infty}E_n^{(k)}(x;\lambda)\frac{t^n}{n!}, \tag{1}$$

($|t|<\pi$ when $\lambda=1$ and $|t|<|\ln(-\lambda)|$ when $\lambda\neq 1$), $\lambda\in\mathbb{C}$, set of complex numbers, $k\in\mathbb{N}=\{1,2,3,\ldots\}$. Setting $x=0$ in (1), we have the first kind Apostol-Euler numbers of higher-order as follows

$$E_n^{(k)}(\lambda)=E_n^{(k)}(0;\lambda).$$

Setting $k=\lambda=1$ in (1), we have the Euler numbers of the first kind

$$E_n=E_n^{(1)}(1)$$

(*cf.* [4]-[40]; see also the references cited in each of these earlier works).

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By using generating function, we give few values of the Euler numbers of the first kind as follows:

$$E_0 = 1, E_1 = -\frac{1}{2}, E_3 = \frac{1}{4}, \ldots$$

Observe that when $n > 0$, it is easy to see that

$$E_{2n} = 0.$$

The second kind Euler numbers are defined by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}$$

(*cf.* [4]-[40]; see also the references cited in each of these earlier works). By using this generating function, we compute few values of the numbers E_n^* as follows:

$$E_0^* = 1, E_2^* = -1, E_4^* = 5, E_6^* = -61, \ldots$$

Observe that for $n \geq 0$, we have

$$E_{2n+1}^* = 0$$

and

$$E_n^* = 2^n E_n\left(\frac{1}{2}\right)$$

(*cf.* [**?**]-[23], [28], [33]; see also the references cited in each of these earlier works).

The second kind Euler numbers of negative order are defined by means of the following generating functions:

$$F_{E2}(t, k) = \left(\frac{2}{e^t + e^{-t}}\right)^{-k} = \sum_{n=0}^{\infty} E_n^{*(-k)} \frac{t^n}{n!}, \tag{2}$$

where $|t| < \frac{\pi}{2}$ (*cf.* [5]-[40]; see also the references cited in each of these earlier works).

The λ-array polynomials $S_v^n(x; \lambda)$ are defined by the following generating function (*cf.* [30]):

$$F_A(t, x, v; \lambda) = \frac{\left(\lambda e^t - 1\right)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} S_v^n(x; \lambda) \frac{t^n}{n!}, \tag{3}$$

where $v \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\lambda \in \mathbb{C}$ (*cf.* see also, [4], [6], [7], [11], [30], [31]; and see also the references cited in each of these earlier works).

In [32], we gave the numbers $y_1(n, k; \lambda)$ and $y_2(n, k; \lambda)$, which are defined by means of the following generating functions, respectively:

$$F_{y_1}(t, k; \lambda) = \frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} \tag{4}$$

and

$$F_{y_2}(t, k; \lambda) = \frac{1}{(2k)!}\left(\lambda e^t + \lambda^{-1} e^{-t} + 2\right)^k = \sum_{n=0}^{\infty} y_2(n, k; \lambda) \frac{t^n}{n!}. \tag{5}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. By using (4) and (5), few values of the numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$ are given as follows, respectively:

$$y_1(0,0;\lambda) = 1, y_1(0,1;\lambda) = \lambda + 1, y_1(0,2;\lambda) = \frac{1}{2}\lambda^2 + \lambda + \frac{1}{2},$$
$$y_1(1,0;\lambda) = 0, y_1(1,1;\lambda) = \lambda, y_1(1,2;\lambda) = \lambda^2 + \lambda,$$
$$y_1(2,0;\lambda) = 0, y_1(2,1;\lambda) = \lambda, y_1(2,2;\lambda) = 2\lambda^2 + \lambda$$

and

$$y_2(0,0;\lambda) = 1, y_2(0,1;\lambda) = \frac{1}{2\lambda} + \frac{\lambda}{2} + \frac{1}{2}, y_2(0,2;\lambda) = \frac{\lambda^2 + 4\lambda}{24} + \frac{\lambda}{24\lambda^2} + \frac{1}{4},$$
$$y_2(1,0;\lambda) = 0, y_2(1,1;\lambda) = \frac{\lambda}{2} - \frac{1}{2\lambda}, y_2(1,2;\lambda) = \frac{\lambda^2 + 2\lambda}{12} - \frac{2\lambda + 1}{6\lambda^2},$$
$$y_2(2,0;\lambda) = 0, y_2(2,1;\lambda) = \frac{\lambda}{2} + \frac{1}{2\lambda}, y_2(2,2;\lambda) = \frac{\lambda^2 + \lambda}{6} + \frac{\lambda + 1}{6\lambda^2}.$$

Replacing λ by $-\lambda$ in (4), then the numbers $y_1(n,k;\lambda)$ reduces to the λ-Stirling numbers $S_2(n,v;\lambda)$, which are defined by means of the following generating function:

$$F_S(t,v;\lambda) = \frac{\left(\lambda e^t - 1\right)^v}{v!} = \sum_{n=0}^{\infty} S_2(n,v;\lambda)\frac{t^n}{n!}, \tag{6}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [25], [30], [39]; see also the references cited in each of these earlier works).

By using (6), we give few values of the numbers $S_2(n,v;\lambda)$ as follows:

$$S_2(0,0;\lambda) = 1, S_2(1,0;\lambda) = 0, S_2(1,1;\lambda) = \lambda, S_2(2,0;\lambda) = 0, S_2(2,1;\lambda) = \lambda, \ldots$$

and

$$S_2(0,v;\lambda) = \frac{(\lambda - 1)^v}{v!}.$$

If we set $\lambda = 1$ in (6), then the numbers $S_2(n,v;\lambda)$ are reduce to the Stirling numbers of the second kind

$$S_2(n,v) = S_2(n,v;1).$$

(*cf.* [3]-[40]; see also the references cited in each of these earlier works).

We ([32], [36]) investigated various properties of the numbers $y_1(n,k;\lambda)$. These numbers are related to the following combinatorial sum:

$$y_1(n,k;1) = \sum_{j=0}^{k} \binom{k}{j} j^n = \frac{d^n}{dt^n}\left(e^t + 1\right)^k |_{t=0}, \tag{7}$$

where $n = 1, 2, \ldots$(*cf.* [12], [32], [36]).

We see that

$$S_2(n,v;\lambda) = (-1)^k y_1(n,k;-\lambda)$$

(*cf.* [32]).

Let a and b are real numbers and λ real or complex numbers. The numbers $y_3(n,k;\lambda;a,b)$ are defined by means of the following generating functions (*cf.* [33]):

$$F_{y_3}(t,k;\lambda;a,b) = \frac{e^{bkt}}{k!}\left(\lambda e^{(a-b)t} + 1\right)^k = \sum_{n=0}^{\infty} y_3(n,k;\lambda;a,b)\frac{t^n}{n!}. \tag{8}$$

Note that there is one generating function for each value of k.

By using (8), we have

$$y_3(n,k;\lambda;a,b) = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j} \lambda^j (bk + j(a-b))^n, \tag{9}$$

(*cf.* [33]).

The central factorial numbers of the first kind, $T(n,k)$ are defined by means of the following generating function:

$$F_T(t,k) = \frac{1}{(2k)!}\left(e^t + e^{-t} - 2\right)^k = \sum_{n=0}^{\infty} T(n,k)\frac{t^{2n}}{(2n)!} \tag{10}$$

(*cf.* [3], [9], [10], [16], [40], [31]; see also the references cited in each of these earlier works). For $n,k \in \mathbb{N}$, we have

$$T(0,k) = T(n,0) = 0.$$

Remark 1. *The central factorial numbers are related to the rook polynomials, which count the number of ways of placing non-attacking rooks on a chess board (cf.* [1]*). In the work of Alayont and Krzywonos* [2]*: the number of ways to place k rooks on a size m triangle board in three dimensions is equal to*

$$T(m+1, m+1-k),$$

where $0 \le k \le m$.

Remark 2. *In* [9, Eq-(3.15)]*, Cigler gave the following formula for the numbers $T(m,k)$:*

$$T(m,k) = \frac{1}{(2k)!}\sum_{j=0}^{2k}(-1)^j \binom{2k}{j} (j-k)^{2m}. \tag{11}$$

The formal power series

$$H(t,k) = \sum_{m=0}^{\infty} T(m,k)\frac{t^{2m}}{(2m)!}$$

is the uniquely determined solution of the following differential equation

$$\left(\frac{e^t-1}{e^t+1}\right)\frac{d}{dt}H(t,k) = kH(t,k)$$

with

$$T(k,k) = 1.$$

Throughout this paper, we use the following notations:

$\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- = \{-1,-2,-3,\ldots\}$. Here, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Furthermore, $0^n = 1$ if $n = 0$, and, $0^n = 0$ if $n \in \mathbb{N}$.

$$\binom{x}{v} = \frac{x(x-1)\cdots(x-v+1)}{v!} = \frac{(x)_v}{v!}$$

(*cf.* [3]-[40]; see also the references cited in each of these earlier works).

2. Identities and relations on new families of numbers and polynomials

In [34], we defined the numbers $y_4(n,k;\lambda;a,b)$ and the polynomials $P_n(x;k;a,b,\lambda)$ by means of the following generating function, respectively:

Let $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$.

(12)

$$F_{y_4}(t,k;\lambda;a,b) = \left(e^{ta}+\lambda e^{bt}\right)^k \sum_{m=0}^{\infty} \binom{-k}{m} (a+b)^m = \sum_{n=0}^{\infty} y_4(n,k;\lambda;a,b)\frac{t^n}{n!}$$

and

$$\mathfrak{F}(t,x,k;\lambda;a,b) = F_{y_4}(t,k;\lambda;a,b)e^{tx} = \sum_{n=0}^{\infty} P_n(x,k;\lambda;a,b)\frac{t^n}{n!}. \tag{13}$$

Note that there is one generating function for each value of k.

By using (13), we have

(14)

$$P_n(x,k;\lambda;a,b) = \frac{1}{(a+b+1)^k}\sum_{j=0}^{k}\sum_{l=0}^{n} \binom{k}{j}\binom{n}{l} \lambda^{k-j} x^{n-l} (ja+(k-j)b)^{n-l}$$

(*cf.* [34]). Substituting $x=0$ into the above equation, we have

$$y_4(n,k;\lambda;a,b) = \frac{1}{(a+b+1)^k}\sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j}(ja+(k-j)b)^n \tag{15}$$

(*cf.* [34]). Partial derivative of the polynomials $P_n(x,k;\lambda,a,b)$ with respect to x is given by

$$\frac{\partial}{\partial x}P_n(x,k;\lambda,a,b) = nP_{n-1}(x,k;\lambda,a,b)$$

(*cf.* [34]).

The polynomials $P_n(x,k;\lambda;a,b)$ are related to following relation:

$$P_n(x,-k;\lambda;1,0) = \lambda^{-k}E_n^{(k)}(x;\lambda)$$

(*cf.* [34]).

By combining (8) with (12), we set the following functional equation, which gives a relationship between the numbers $y_3(n,k;\lambda;a,b)$ and the numbers $y_4(n,k;\lambda;a,b)$:

$$F_{y_3}(t,k;\lambda^{-1};a,b) = \frac{\lambda^k}{k!}e^{kt\left(\frac{a+b}{2}\right)}F_{y_4}\left(t,k;\lambda;\frac{a-b}{2},\frac{b-a}{2}\right).$$

By using the above functional equation, we get

$$\sum_{n=0}^{\infty} y_3(n,k;\lambda^{-1};a,b)\frac{t^n}{n!} = \frac{\lambda^k}{k!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\left(k\frac{a-b}{2}\right)^{n-j} y_4\left(j,k;\lambda;\frac{a-b}{2},\frac{b-a}{2}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$on both sides of the above equation, we arrive at the following theorem:

Theorem 2.1.

$$y_3(n,k;\lambda;a,b)=\frac{\lambda^k}{k!}\sum_{j=0}^{n}\binom{n}{j}\left(k\frac{a-b}{2}\right)^{n-j}y_4\left(j,k;\lambda;\frac{a-b}{2},\frac{b-a}{2}\right).$$

By combining (12) with (10), we set the following functional equation:

$$F_{y_4}\left(t,k;1;1;\frac{a-b}{2},\frac{b-a}{2}\right)=\sum_{j=0}^{k}\binom{k}{j}(2j)!2^{k-j}F_T\left(\frac{a-b}{2}t,j\right).$$

By the above equation, we obtain

$$\sum_{n=0}^{\infty}y_4\left(n,k;1;\frac{a-b}{2},\frac{b-a}{2}\right)\frac{t^n}{n!}=\sum_{j=0}^{k}\binom{k}{j}(2j)!2^{k-j}\sum_{n=0}^{\infty}T(n,j)\frac{\left(\frac{a-b}{2}t\right)^{2n}}{(2n)!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$on both sides of the above equation, we arrive at following theorem including a relation between the numbers $y_4\left(n,k;1;a,b\right)$ and the numbers $T\left(n,k\right)$:

Theorem 2.2.

$$y_4\left(2n,k;1;\frac{a-b}{2},\frac{b-a}{2}\right)=(a-b)^{2n}\sum_{j=0}^{k}\binom{k}{j}(2j)!2^{k-2n-j}T\left(n,j\right).$$

By combining (12) with (5), we obtain the following functional equation:

$$F_{y_4}\left(t,k;1;\lambda;\frac{a-b}{2},\frac{b-a}{2}\right)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(2j)!2^{k-j}F_{y_2}\left(\frac{a-b}{2}t,k;1\right).$$

By using the above functional equation, we have

$$\sum_{n=0}^{\infty}y_4\left(n,k;1;\frac{a-b}{2},\frac{b-a}{2}\right)\frac{t^n}{n!}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(2j)!2^{k-j}\sum_{n=0}^{\infty}y_2(n,k;1)\frac{\left(\frac{a-b}{2}t\right)^{2n}}{(2n)!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$on both sides of the above equation, we arrive at following theorem including a relation between the numbers $y_4\left(n,k;1;a,b\right)$ and $y_2(n,k;1)$:

Theorem 2.3.

$$\begin{aligned}&y_4\left(2n,k;1;\frac{a-b}{2},\frac{b-a}{2}\right)\\ =&\ \frac{(a-b)^{2n}}{(a+b+1)^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(2j)!2^{k-2n-j}y_2(n,j;1).\end{aligned}$$

By using (4) and (12), we get the following equation:

$$F_{y_4}\left(t,k;\lambda;a,b\right)=\frac{\lambda^k e^{bkt}k!}{(a+b+1)^k}F_{y_1}\left((a-b)t,k;\lambda^{-1}\right). \tag{16}$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} y_4(n,k;\lambda;a,b)\frac{t^n}{n!}$$
$$= \frac{\lambda^k k!}{(a+b+1)^k}\sum_{n=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(a-b)^j(bk)^{n-j}y_1(j,k;\lambda^{-1}).$$

Comparing the coefficients of $\frac{t^n}{n!}$on both sides of the above equation, we give a relation between the numbers $y_4(n,k;\lambda;a,b)$ and the numbers $y_1(n,k;\lambda)$ by the following theorem:

Theorem 2.4.
(17)
$$y_4(n,k;\lambda;a,b) = \frac{\lambda^k k!}{(a+b+1)^k}\sum_{j=0}^{k}\binom{k}{j}(a-b)^j(bk)^{n-j}y_1(j,k;\lambda^{-1}).$$

We modify equation (16), we get

$$F_{y_4}(t,k;\lambda;a,b) = \frac{\lambda^k k!}{(a+b+1)^k}F_{y_3}\left(t,k;\lambda^{-1}:a,b\right).$$

By using the above functional equation, we get

$$\sum_{n=0}^{\infty} y_4(n,k;\lambda;a,b)\frac{t^n}{n!} = \frac{\lambda^k k!}{(a+b+1)^k}\sum_{n=0}^{\infty} y_3(n,k;\lambda^{-1};a,b)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$on both sides of the above equation, we arrive at the following theorem:

Theorem 2.5.

(18) $$y_4(n,k;\lambda;a,b) = \frac{\lambda^k k!}{(a+b+1)^k}y_3(n,k;\lambda^{-1};a,b).$$

Combining (17) and (18), we get the following corollary:

Corollary 2.6.

(19) $$y_3(n,k;\lambda^{-1};a,b) = \sum_{j=0}^{k}\binom{k}{j}(a-b)^j(bk)^{n-j}y_1(j,k;\lambda^{-1}).$$

Proof of (19) was also given in [33, Theorem 5].
By using equation (16), we also obtain the following corollary:

Corollary 2.7.

$$y_4(n,k;\lambda;a,b) = \frac{(2\lambda)^k(a-b)^n}{(a+b+1)^k}E_n^{(-k)}(bk;\lambda^{-1}).$$

Combining (3) and (12), we obtain the following functional equation:

$$F_{y_4}(t,k;-\lambda;a,b) = \frac{\lambda^k k!}{(a+b+1)^k}F_A\left((a-b)t,\frac{bk}{a-b},k;\lambda^{-1}\right).$$

By using the above equation, we obtain

$$\sum_{n=0}^{\infty} y_4(n,k;-\lambda;a,b)\frac{t^n}{n!}$$
$$= \frac{\lambda^k k!}{(a+b+1)^k}\sum_{n=0}^{\infty} S_k^n\left(\frac{bk}{a-b};\lambda^{-1}\right)(a-b)^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$on both sides of the above equation, we give a relation between the numbers $y_4(n,k;\lambda;a,b)$ and the array polynomials $S_k^n(x;\lambda)$ by the following theorem:

Theorem 2.8.

$$y_4(n,k;-\lambda;a,b) = \frac{(a-b)^n\lambda^k k!}{(a+b+1)^k}S_k^n\left(\frac{bk}{a-b};\lambda^{-1}\right).$$

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